Explaining Hadamard with quantum mechanics

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

eigenvalues $= \pm 1$

eigen vectors (normalized) =

$$\frac{1}{\sqrt{4-2\sqrt{2}}} \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} \quad \& \quad \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{pmatrix} 1 \\ -\sqrt{2}-1 \end{pmatrix}$$

Now, the Hamiltonian is $\hat{H}$

$$H = e^{-i \frac{\hat{H}}{\hbar} t}$$

The eigen values of H are $e^{-i \frac{\lambda t}{\hbar}}$

where λ are the eign values of $\hat{H}$

thus $1 \rightarrow e^{-i \lambda_1 t/\hbar} \Rightarrow \lambda_1 = 0$

$-1 \rightarrow e^{-i \lambda_2 t/\hbar} \Rightarrow \lambda_2 = \pi h$

assuming a unit of time

-side

$$\boxed{\begin{aligned} 1+(-\sqrt{2}-1)^2 &= 1+2+1+2\sqrt{2} \\ &= 4+2\sqrt{2} \\ 1+(\sqrt{2}-1)^2 &= 1+2+1-2\sqrt{2} \\ &= 4-2\sqrt{2} \end{aligned}}$$

$$c_1 = \langle v_1^* | \psi \rangle$$

$$= \frac{1}{\sqrt{4-2\sqrt{2}}} \begin{bmatrix} 1 & \sqrt{2}-1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\boxed{c_1 = \frac{1}{\sqrt{4-2\sqrt{2}}} \left(\alpha + (\sqrt{2}-1)\beta \right)}$$

$$c_2 = \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{bmatrix} 1 & -\sqrt{2}-1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\boxed{c_2 = \frac{1}{\sqrt{4+2\sqrt{2}}} \left(\alpha - (\sqrt{2}+1)\beta \right)}$$

$$\psi(t) = \sum_{n=1}^{2} c_n \psi_n e^{-iE_n t/\hbar}$$

$$= c_1 \psi_1 + c_2 \psi_2 e^{-i\pi t}$$

$$= \frac{1}{4-2\sqrt{2}}\left(\alpha+\beta(\sqrt{2}-1)\right)\begin{bmatrix}1\\ \sqrt{2}-1\end{bmatrix} +$$

$$\frac{1}{4+2\sqrt{2}}\left(\alpha+\beta(-\sqrt{2}-1)\right)\begin{bmatrix}1\\ -\sqrt{2}-1\end{bmatrix} e^{-i\pi t}$$

$t=0$

$$= \frac{(4+2\sqrt{2})(\alpha+\beta(+\sqrt{2}-1))}{(4-2\sqrt{2})(4+2\sqrt{2})}\begin{bmatrix}1\\ \sqrt{2}-1\end{bmatrix} +$$

$$\frac{(4-2\sqrt{2})(\alpha+\beta(-\sqrt{2}-1))}{(4+2\sqrt{2})(4-2\sqrt{2})}\begin{bmatrix}1\\ -\sqrt{2}-1\end{bmatrix}$$

$$= 2\times\frac{(\overset{2}{\cancel{4}}+\overset{1}{\cancel{2}}\sqrt{2})\alpha + (\overset{2}{\cancel{4}}+\overset{1}{\cancel{2}}\sqrt{2})(\sqrt{2}-1)\beta}{16-8}\begin{bmatrix}1\\ \sqrt{2}-1\end{bmatrix}$$

$$+\, 2\times\frac{(\overset{2}{\cancel{4}}-\overset{1}{\cancel{2}}\sqrt{2})\alpha + \beta(-\sqrt{2}-1)(\overset{2}{\cancel{4}}-\overset{1}{\cancel{2}}\sqrt{2})}{8}\begin{bmatrix}1\\ -\sqrt{2}-1\end{bmatrix}$$

$$= \frac{1}{4}\left[(2+\sqrt{2})\alpha + (2\sqrt{2}-2+2-\sqrt{2})\beta\right)\begin{bmatrix}1\\ \sqrt{2}-1\end{bmatrix}$$

$$+\, \frac{1}{4}\left((2-\sqrt{2})\alpha + (-2\sqrt{2}+2-2+\sqrt{2})\beta\right.\begin{bmatrix}1\\ -\sqrt{2}-1\end{bmatrix}$$

$$= \frac{1}{4}\left((2+\sqrt{2})\alpha + (\sqrt{2})\beta \right) \begin{bmatrix} 1 \\ \sqrt{2}-1 \end{bmatrix}$$

$$+ \frac{1}{4}\left((2-\sqrt{2})\alpha + (-\sqrt{2}\beta) \right) \begin{bmatrix} 1 \\ -\sqrt{2}-1 \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4\alpha \\ (2+\sqrt{2})(\sqrt{2}-1)\alpha + \sqrt{2}(\sqrt{2}-1)\beta \; + \\ -(2-\sqrt{2})(\sqrt{2}+1)\alpha + \sqrt{2}(\sqrt{2}+1)\beta \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4\alpha \\ (2\sqrt{2}-2+2-\sqrt{2})\alpha + (2-\sqrt{2})\beta \; + \\ -(2\sqrt{2}+2-2-\sqrt{2})\alpha + (2+\sqrt{2})\beta \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4\alpha \\ \cancel{\sqrt{2}\alpha} + (2-\sqrt{2})\beta - \cancel{(\sqrt{2})\alpha} + (2+\sqrt{2})\beta \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 4\alpha \\ 4\beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

at $t = 1$ $\quad e^{-i\pi t} \Rightarrow -1$

$$\psi(1) = \frac{1}{4}\left((2+\sqrt{2})\alpha + (\sqrt{2})\beta \right) \begin{bmatrix} 1 \\ \cdots \end{bmatrix}$$

$$\sqrt{2}-1$$

$$+\frac{1}{4}\left((2-\sqrt{2})\alpha+(-\sqrt{2}\beta)\right)\begin{bmatrix}1\\-\sqrt{2}-1\end{bmatrix} \times -1$$

$$=\frac{1}{4}\begin{bmatrix}2\sqrt{2}\alpha+2\sqrt{2}\beta\\ \\(2+\sqrt{2})(\sqrt{2}-1)\alpha+(\sqrt{2})(\sqrt{2}-1)\beta+\\+(\sqrt{2}+1)(2-\sqrt{2})\alpha-\sqrt{2}(\sqrt{2}+1)\beta\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}2\sqrt{2}\alpha+2\sqrt{2}\beta\\(2\sqrt{2}-2+2-\sqrt{2})\alpha+(2-\sqrt{2})\beta+\\(2\sqrt{2}-2+2-\sqrt{2})\alpha-(2+\sqrt{2})\beta\end{bmatrix}$$

$$=\frac{1}{4}\begin{bmatrix}2\sqrt{2}\alpha+2\sqrt{2}\beta\\\sqrt{2}\alpha+(2-\sqrt{2})\beta+\sqrt{2}\alpha-(2+\sqrt{2})\beta\\=2\sqrt{2}\alpha-2\sqrt{2}\beta\end{bmatrix}$$

$$=\frac{1}{2}\begin{bmatrix}\sqrt{2}\alpha+\sqrt{2}\beta\\\sqrt{2}\alpha-\sqrt{2}\beta\end{bmatrix}=\frac{\sqrt{2}}{2}\begin{bmatrix}\alpha+\beta\\\alpha-\beta\end{bmatrix}$$

$$=\frac{1}{\sqrt{2}}\begin{bmatrix}\alpha+\beta\\\alpha-\beta\end{bmatrix}$$